JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 32, No. 2, March–April 2009

Engineering Notes

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Numerical State-Dependent Riccati Equation Approach for Missile Integrated Guidance Control

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DOI: 10.2514/1.34291

Nomenclature

$A(x)$, $B(x)$	=	state-dependent system matrices of size $n \times n$ and $n \times m$
$Q(x)$, $R(x)$	=	state-dependent weighting matrices of sizes $n \times n$ and $m \times m$
r, $\dot{r}$	=	range and range rate of the target with respect to the missile
S	=	solution to the Riccati equation
T	=	rocket motor thrust per unit mass acting along the longitudinal axis of the missile
u	=	control vector of size $m \times 1$
upert	=	control perturbation vector of size $m \times 1$
x	=	state vector of size $n \times 1$
xpert	=	state perturbation vector of size $n \times 1$
ΔX, ΔY, ΔZ	=	relative position components of the target with respect to the missile
δ_y, δ_z	=	position of the moving masses along the pitch and yaw axes with respect to the body
δ_{yc}, δ_{zc}	=	moving-mass position commands
θ, ψ	=	pitch and yaw Euler angles of the missile
λ_y, λ_z	=	line-of-sight angles

I. Introduction

INTEGRATED synthesis of missile guidance and control systems has been of significant interest in the recent literature [1–6]. These techniques have been shown to enhance missile performance by exploiting the synergism between guidance and control (autopilot) subsystems. By establishing additional feedback paths in the flight control system, integrated design methods allow the designer to exploit beneficial interactions between these subsystems. A more detailed discussion of traditional and integrated guidance control of missiles is available in [6].

State-dependent Riccati equation (SDRE) methodology for nonlinear control system design problems is being actively pursued for applications in different fields [1,2,7–11]. The advantage of SDRE is that it is a nonlinear control technique that allows the designer tools that are very similar to the linear quadratic regulator (LQR). The SDRE approach has been used for integrated guidance control of missiles in [1,2]. Missile longitudinal autopilot has been designed using this approach in [11]. An overview of the approach has been presented in [8]. The first step in this methodology is that of deriving a state-dependent coefficient (SDC) parameterization of the system dynamics. This is typically achieved by analytical manipulation of the nonlinear vector field terms governing the dynamics of the system. The analytical approach is not suitable for high-dimensional systems and systems with dynamics provided in numerical form. To overcome these limitations a novel numerical approach is used in this work. The algorithm is a modified version of a previously derived version in [7] by the second and third authors of this paper.

The focus of the present work is the development of a numerical approach to integrated guidance control formulation for a moving-mass-actuated kinetic warhead using state-dependent Riccati equation methodology. The SDRE technique is briefly discussed in Sec. II. Numerical SDC parameterization algorithm is developed in Sec. III. Integrated guidance control methodology for moving-mass-actuated missiles is discussed in Sec. IV. Closed-loop simulation results are presented in Sec. V.

II. SDRE Controller Design

A nonlinear dynamic system described by Eq. (1) is considered:

$$\dot{x} = f(x, u) \tag{1}$$

where f is a $(n \times 1)$ vector. It is assumed that the right-hand side of the preceding equation is smooth, continuous, and satisfies the requirement that $f(0,0) = 0$. As the first step in the SDRE design process, the equations of motion are cast in the SDC form:

$$\dot{x} = A(x)x + B(x)u \tag{2}$$

The control problem is formulated as the minimization of a state-dependent quadratic performance index described by Eq. (3). Note the state dependence of the state and control weighting matrices $Q(x)$ and $R(x)$:

$$J = \frac{1}{2}\int_0^\infty (x^T Q(x)x + u^T R(x)u) \tag{3}$$

The resulting feedback controller [8] can be shown to be

$$u = -R(x)^{-1}B(x)^T S(x)x \tag{4}$$

where $S(x)$ is the solution to the state-dependent algebraic Riccati equation:

$$A(x)S(x) + S(x)A^T(x) - S(x)B^T(x)R^{-1}(x)B(x)S(x) + Q(x) = 0 \tag{5}$$

Presented as Paper 6672 at the AIAA Guidance, Navigation, and Control Conference, Hilton Head, SC, 20–23 August 2007; received 27 August 2007; revision received 23 September 2008; accepted for publication 23 September 2008.

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Note that the formulation is very similar to the well-known LQR problem. However, unlike the LQR problem, the gain is not a constant and varies as a function of the state x. At a given value of the state x, the state-dependent algebraic Riccati equation can be solved using numerical techniques.

III. Numerical Technique for SDC Parameterization

The first step in the SDRE control system design process is to obtain a representation of the system dynamics as shown in Eq. (6):

$$\dot{x} = f(x,u) = A(x)x + B(x)u \tag{6}$$

A numerical technique for evaluating the A and B matrices for a given value of x will be developed in this section. Any $n \times n$ matrix that satisfies Eq. (7) for a given value of x is a candidate solution for A:

$$Ax = f(x,0) \tag{7}$$

The preceding system of equations for A is underdetermined and can therefore have infinite solutions. However, the extra degrees of freedom could be used to construct an A matrix that varies smoothly with x. This is achieved by enforcing Eq. (7) for perturbed state vectors, which are created by perturbing a single component of the state vector at a time. The perturbed state vectors are represented as

$$\begin{aligned} x_{i+\varepsilon} &= x + \begin{bmatrix} 0 & 0 & \cdots & \text{xpert}_i & \cdots & 0 \end{bmatrix}^T \\ x_{i-\varepsilon} &= x + \begin{bmatrix} 0 & 0 & \cdots & -\text{xpert}_i & \cdots & 0 \end{bmatrix}^T, \qquad i = 1,\ldots,n \end{aligned} \tag{8}$$

where xpert_i represents a small perturbation of the ith component of the state vector. The A matrix is held fixed for small perturbations of the state vector around the current value. Enforcing Eq. (7) for perturbed vectors given by Eq. (8) results in

$$Ax = f(x,0), \qquad Ax_{i\pm\varepsilon} = f(x_{i\pm\varepsilon},0), \qquad i = 1,\ldots,n \tag{9}$$

Using a column vector representation $a = A^T(:)$, Eq. (8) can be rewritten as

$$Xa = f(x,0), \qquad X_{i\pm\delta}a = f(x_{i\pm\delta},0), \qquad i = 1,\ldots,n \tag{10}$$

where

$$X = \begin{bmatrix} x^T & 0 & 0 & 0 & 0 \\ 0 & x^T & & & 0 \\ 0 & & \ddots & & \vdots \\ 0 & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & x^T \end{bmatrix}$$
$$X_{i\pm\varepsilon} = \begin{bmatrix} x_{i\pm\varepsilon}^T & 0 & 0 & 0 & 0 \\ 0 & x_{i\pm\varepsilon}^T & & & 0 \\ 0 & & \ddots & & \vdots \\ 0 & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & x_{i\pm\varepsilon}^T \end{bmatrix}, \qquad i = 1,\ldots,n \tag{11}$$

The $2n+1$ subequations in Eq. (10) could be stacked into one single equation for the A matrix as

$$\tilde{X}_{(n+2n^2)\times n^2}\, a = F_{(n+2n^2)\times 1} \tag{12}$$

where

$$\tilde{X} = \begin{bmatrix} X_{1+\varepsilon} \\ X_{2+\varepsilon} \\ \vdots \\ X_{n+\varepsilon} \\ X_{1-\varepsilon} \\ X_{2-\varepsilon} \\ \vdots \\ X_{n-\varepsilon} \\ X \end{bmatrix}_{(n+2n^2)\times n^2}, \qquad F = \begin{bmatrix} f(x_{1+\varepsilon},0) \\ f(x_{2+\varepsilon},0) \\ \vdots \\ f(x_{n+\varepsilon},0) \\ f(x_{1-\varepsilon},0) \\ f(x_{2-\varepsilon},0) \\ \vdots \\ f(x_{n-\varepsilon},0) \\ f(x,0) \end{bmatrix}_{(n+2n^2)\times 1} \tag{13}$$

The preceding set of equations for the A matrix is now overdetermined. A least-squares minimization solution to the system of equations can be obtained. The A matrix is typically not completely populated with nonzero entries. Zero entries in the A matrix can be established by the evaluation of the system dynamics with the perturbed state vectors. The element a_{ij} is set to zero if the perturbation of the jth state component alone does not create a change in ith component of f. This information can be posed as a constraint in the least-squares optimization problem:

$$K_{k\times n^2}\, a = 0_{k\times 1} \tag{14}$$

where k is the total number of zero entries in the A matrix. The K matrix consists of only zeros and ones. Each row of the K matrix has a 1 corresponding to a 0 entry in the A matrix. It should be noted that these entries are not constant and are dependent on the current value of x. The A matrix is finally obtained as solution to the following constrained optimization problem:

$$\min(\tilde{X}a - F)^T(\tilde{X}a - F) \quad \text{subject to } Ka = 0 \tag{15}$$

The solution to the preceding minimization problem can be written as

$$\begin{bmatrix} a \\ p \end{bmatrix} = \begin{bmatrix} 2\tilde{X}^T\tilde{X} & K^T \\ K & 0 \end{bmatrix}^{\#} \begin{bmatrix} 2\tilde{X}^T F \\ 0 \end{bmatrix} \tag{16}$$

where # represents the pseudoinverse operator, and p represents the constraint Lagrange multiplier.

The computational procedure for the B matrix is much simpler and more accurate compared with the A matrix. It is assumed that the control appears linearly in $f(x,u)$. Therefore, the columns of the B matrix that are equal to the number of controls can be computed exactly by perturbing one control at a time:

$$B(:,i) = \frac{f(x,u_{i+\varepsilon}) - f(x,0)}{\text{upert}_i} \tag{17}$$

where $u_{i+\varepsilon} = [0\ 0\ \cdots\ \text{upert}_i\ \cdots\ 0]^T$.

IV. Integrated Guidance Control

The numerical SDRE technique is applied to the integrated guidance control of a moving-mass-actuated missile [5] in this section. The missile flight control objective is the interception of a ballistic target using a moving-mass actuation system. To pose the target interception problem as a nonlinear regulation problem suitable for SDRE approach, a set of state variables has to be first identified. These states would consist of guidance states, missile attitude states, and missile actuator states. Guidance states to achieve the objective of target interception have been identified as line-of-sight rates in [5]. The missile under consideration has only two controls: one along the pitch and one along the yaw axes. Therefore, the roll channel cannot be controlled. Attitude states such as Euler angles and the body rates are not suitable for SDC parameterization because they do not satisfy the requirement $f(0,0) = 0$. To find the set of states that satisfy the requirement $f(0,0) = 0$, the line-of-sight dynamics is further analyzed. Line-of-sight angles of the target with respect to the missile can be expressed as follows:

$$\tan\lambda_y = \frac{\Delta Y}{\Delta X}, \qquad \tan\lambda_z = \frac{-\Delta Z}{\sqrt{\Delta X^2 + \Delta Y^2}} \tag{18}$$

Differentiating Eq. (18) twice, the model for line-of-sight rate dynamics can be written as shown in Eqs. (19) and (20):

$$\ddot{\lambda}_y = \frac{\cos\lambda_y \Delta\ddot{Y} - \sin\lambda_y \Delta\ddot{X}}{r_{xy}} - \dot{\lambda}_y \frac{2\dot{r}_{xy}}{r_{xy}} \tag{19}$$

$$\ddot{\lambda}_z = \frac{(-\cos\lambda_z \Delta\ddot{Z} - \sin\lambda_z(\cos\lambda_y \Delta\ddot{X} + \sin\lambda_y \Delta\ddot{Y}))}{r} - \frac{\sin\lambda_z r_{xy}\dot{\lambda}_y^2}{r} - \frac{2\dot{\lambda}_z \dot{r}}{r} \tag{20}$$

The only external force acting on the target is gravity, and the external forces acting on the missile are gravity and constant axial rocket motor thrust. Therefore, the relative acceleration vector can be written as

$$\begin{bmatrix} \Delta\ddot{X} \\ \Delta\ddot{Y} \\ \Delta\ddot{Z} \end{bmatrix} = \begin{bmatrix} -Tc_\psi c_\theta \\ -Ts_\psi c_\theta \\ Ts_\theta \end{bmatrix} \tag{21}$$

where $c(.)$ and $s(.)$ refer to the sin(.) and cos(.) operators. Substituting Eq. (21) in Eq. (19),

$$\ddot{\lambda}_y = \frac{T\cos\theta\sin(\psi - \lambda_y)}{r_{xy}} - \dot{\lambda}_y \frac{2\dot{r}_{xy}}{r_{xy}} \tag{22}$$

From the preceding expression for $\ddot{\lambda}_y$ it is clear that to regulate $\dot{\lambda}_y$ it is necessary to also regulate $(\psi - \lambda_y)$. Repeating the procedure for $\ddot{\lambda}_z$,

$$\ddot{\lambda}_z = \frac{T(-\cos\lambda_z \sin\theta + \sin\lambda_z \cos\theta\cos(\psi - \lambda_y))}{r} - \frac{\sin\lambda_z r_{xy}\dot{\lambda}_y^2}{r} - \frac{2\dot{\lambda}_z \dot{r}}{r} \tag{23}$$

Setting $\psi - \lambda_y = 0$, $\dot{\lambda}_y = 0$, and $\dot{\lambda}_z = 0$, then

$$\ddot{\lambda}_z = \frac{T\sin(\lambda_z - \theta)}{r} \tag{24}$$

From Eq. (24) it can be concluded that to regulate the line-of-sight rates it is necessary to regulate the dynamics of τ_y and τ_z, defined as $\tau_y = \psi - \lambda_y$ and $\tau_z = \theta - \lambda_z$. In other words, the longitudinal axis of the missile along which the constant thrust vector is acting has to point along the line-of-sight (LOS) vector with respect to the target. The Euler angles should approach the LOS angles in steady state $\tau_y \to 0 \Rightarrow \psi \to \lambda_y$ and $\tau_z \to 0 \Rightarrow \theta \to \lambda_z$.

A simulation consisting of the full-fledged nonlinear equations of motion governing the position, attitude, and moving-mass dynamics of the missile based on [5] is used for the evaluation of the SDRE-based integrated guidance controller. A free-falling 3-degree-of-freedom model is assumed for the target. Position commands to the masses are treated as controls. The state vector of interest for the purpose of integrated guidance and control is identified as

$$x = \begin{bmatrix} \dot{\lambda}_y & \dot{\lambda}_z & \tau_y & \tau_z & \dot{\tau}_y & \dot{\tau}_z & \delta_y & \delta_z & \dot{\delta}_y & \dot{\delta}_z \end{bmatrix}^T \tag{25}$$

The last four states in the preceding expression are the actuator states along the pitch and yaw axes. Explicit analytical use of the equations of motion in [5] of the target and missile is never made in the controller design. A C-function that returns the derivatives of the state components in Eq. (25) for a given value of the state vector is developed. This function is referred to as the design model and it contains the full-fledged dynamics of the missile and the target. The state vector alone is not sufficient to evaluate the derivatives of these state components. Auxiliary information such as roll angle, roll rate, range with respect to the target, and line-of-sight angles is also necessary.

The numerical SDC parameterization algorithm discussed in Sec. III and the user-defined inputs xpert and upert are used to compute the A and B matrices in the following equation:

$$\begin{bmatrix} \ddot{\lambda}_y \\ \ddot{\lambda}_z \\ \dot{\tau}_y \\ \dot{\tau}_z \\ \ddot{\tau}_y \\ \ddot{\tau}_z \\ \dot{\delta}_y \\ \dot{\delta}_z \\ \ddot{\delta}_y \\ \ddot{\delta}_z \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \dot{\lambda}_y \\ \dot{\lambda}_z \\ \tau_y \\ \tau_z \\ \dot{\tau}_y \\ \dot{\tau}_z \\ \delta_y \\ \delta_z \\ \dot{\delta}_y \\ \dot{\delta}_z \end{bmatrix} + \begin{bmatrix} \\ B \\ \\ \end{bmatrix} \begin{bmatrix} \delta_{yc} \\ \delta_{zc} \end{bmatrix} \tag{26}$$

Control is then computed using Eqs. (4) and (5) after making a desired choice of the state and control weighting matrices Q and R, respectively. Again, this is also done numerically by using an algebraic Riccati equation solver. The position commands thus computed are saturated to be within the geometric limits of the missile. A position servo is employed to track these position commands. Force applied on the y and z masses to track these commands is computed using the k_p and k_v gains as $F = -k_p(\delta - \delta_c) - k_v\dot{\delta}$. Another saturation function is used to keep the force within the limits before implementing in the simulation. Thus, the controller is implemented on a plant that is much more demanding than the design model.

V. Closed-Loop Simulation Results

Control design parameters used in the closed-loop simulation are given next:

$$\text{xpert} = \begin{bmatrix} 1e-5; & 1e-5; & 1e-2; & 1e-2; & 1e-2; \\ 1e-2; & 1e-2; & 1e-2; & 1e-2; & 1e-2 \end{bmatrix}$$

$$\text{upert} = \begin{bmatrix} 1; & 1 \end{bmatrix}$$

$$Q = \text{diag}([1e8; 1e8; 1e4; 1e4; 1; 1; 1; 1; 1; 1])$$

$$R = \text{diag}([1e4; 1e4])$$

$$\text{Position servo gains } k_p = 26 \quad \text{and} \quad k_v = 11.7$$

Initial conditions for engagement scenario 1 are given in Table 1. Shown in Figs. 1 and 2 are the horizontal-plane and vertical-plane trajectories, respectively, of the missile and the target. The missile successfully intercepts the target with a miss distance of 0.00045 ft that is less than the diameter of the missile. Shown in Figs. 3 and 4 is the convergence of the pitch and the yaw angles to their respective LOS angles. It should be noted that the controller successfully handles large initial condition errors in both the attitude states. Time histories of the y and z actuator mass positions, both actual and commanded, are shown in Figs. 5 and 6, respectively.

Table 1 Initial conditions of missile and target for scenario 1

	North position, ft	East position, ft	Altitude, ft	Velocity, ft/s	Flight-path angle, deg	Heading angle, deg
Target	−41, 140	23,400	241,200	4052	−24.6	−0.031
Missile	0	0	207,800	1993	1.8740	38.27

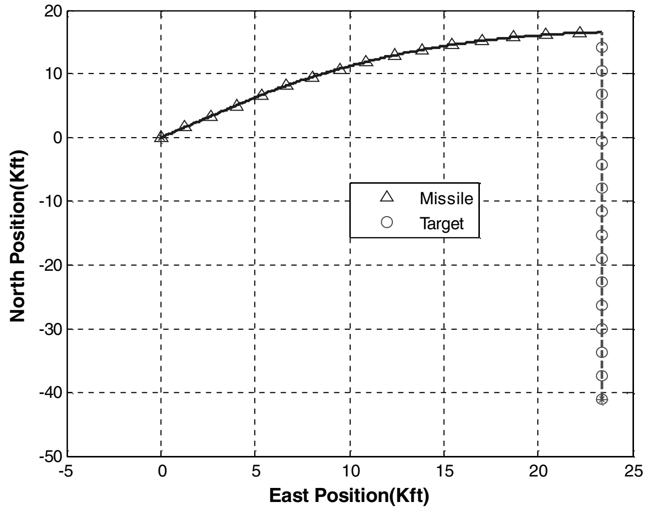

Fig. 1 Horizontal-plane trajectories.

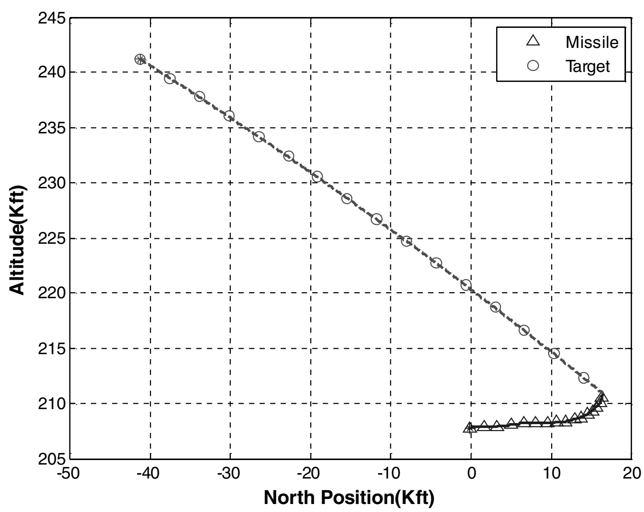

Fig. 2 Vertical-plane trajectories.

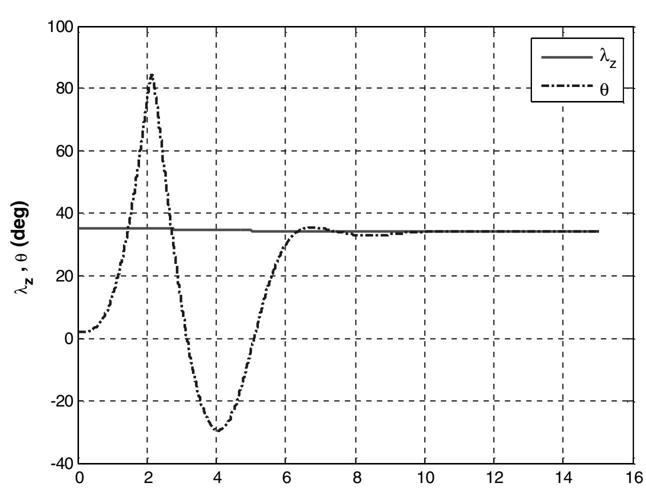

Fig. 3 Pitch-angle time history.

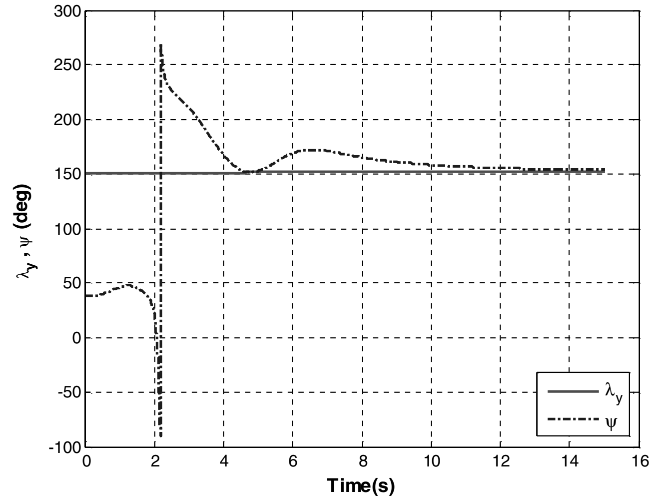

Fig. 4 Yaw-angle time history.

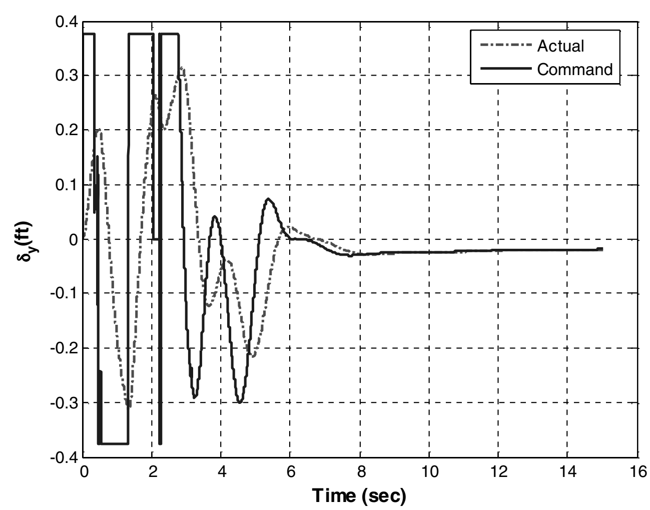

Fig. 5 *Y*-actuator mass position.

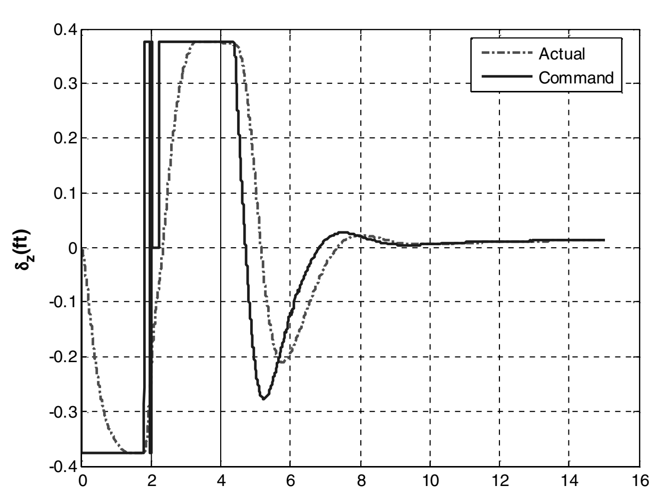

Fig. 6 *Z*-actuator mass position.

VI. Conclusions

A fully numerical approach for implementing controllers based on the state-dependent Riccati equation solution is developed. State-dependent system matrices are obtained as the solution to a constrained least-squares optimization problem. The approach has been applied to the integrated guidance control of a moving-mass-actuated missile. Target interception outside the atmosphere is posed as a tenth-order nonlinear regulation problem and control computation is done using the numerical state-dependent Riccati equation approach discussed in this paper. The effectiveness of the controller is demonstrated in closed-loop simulations with miss distances that were much less than the diameter of the missile.

Acknowledgment

This research was supported under Missile Defense Agency, Small Business Technology Transfer contract HQ0006-05-C-7261.

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